

Asymptotic Expansions for Resonances in the Presence of Small Anisotropic Imperfections

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Abstract

We provide a rigorous derivation of an asymptotic formula for perturbations in the resonance values caused by the presence of finite number of anisotropic imperfections of small shapes with constitutive parameters different from the background conductivity. The asymptotic expansion is carried out with respect to the size of the imperfections. The main feature of the method is to yield a robust procedure making it possible to recover information about the location, shape, and material properties of the anisotropic imperfections.

Key words. resonances, asymptotic expansion, small perturbation, integral operators, anisotropic imperfections

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1 Introduction

Throughout this paper we consider two or three-dimensional bounded domain Ω , and assume we have a smooth background conductivity with small (anisotropic) imperfections of bounded conductivity. The geometry of the imperfections will take the form of ϵB where B is some bounded smooth domain. Our goal is to find an asymptotic expansion for the resonance values of such a domain, with the intention of using the expansion as an aid in identifying the imperfections. That is, we would like to find a method for determining the locations and/or shape of small imperfections by taking resonance measurements.

Theoretical modeling of the interaction between harmonic waves and anisotropic objects is of interest for many physical applications, such as waveguide optics, nondestructive testing, and remote sensing.

Here we discuss the resonances of the transmission problem, which has importance in non-destructive testing of anisotropic materials, in the whole of \mathbb{R}^n , $n = 2, 3$, where the conductivity is assumed to take a positive constant value outside the domain Ω . The transmission eigenvalue problem is a nonlinear and non-selfadjoint eigenvalue problem that is not covered by the standard theory of eigenvalue problems for elliptic equations. The eigenvalue problem with the Dirichlet boundary condition (the Neumann problem, in isotropic media, was treated in [3]), and the scattering problem in $\mathbb{R}^n \setminus \Omega$ with the Dirichlet or the Neumann boundary condition are of equal interest. The asymptotic results for the eigenvalues or the resonances in such cases can be obtained with only minor modifications of the techniques presented here and in [3, 11, 18],

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while the rigorous derivations of similar asymptotic formulae for the scattering problems for the electromagnetic waves (the full Maxwell's equations) or for the Stokes equations require further work.

In this paper we concentrate on deriving rigorously the asymptotic expansion for resonances. This work is considerably different from that in [1, 3, 6, 7, 11, 18] for the eigenvalue problem. In [1, 18], we combined the expansions derived in [9] and an additional lemma with a theorem by Osborn [21] about the convergence of eigenvalues of a sequence of compact operators.

The novelty of this paper, is that it leads to analysis of resonance problems in the presence of multiple anisotropic imperfections. By referring to previous works, the resonance problem is more difficult because the resonance values are not the eigenvalues of a set of compact operators. They can, however, be viewed as singular values of a sequence of meromorphic operator functions. This is achieved by rewriting the problem in terms of integral equations on the boundary of the domain. Then, by using complex operator theory, we derive a formula for the convergence of the resonance values which is in the spirit of the theorem of Osborn. This then leads to an asymptotic expansion for the resonances which is similar to that for the eigenvalues.

The leading order term in the asymptotic expansion contains information about the location, shape, and material properties of the inhomogeneities.

The paper is organized as follows. In Section 2, we model the search of resonances of the transmission problem in the whole of \mathbb{R}^n as two linear spectral problems (2.3) and (2.4). Then, in Section 3, we reformulate (2.3) and (2.4) as two systems of integral equations depending on the small parameter ϵ which is the scale factor associated to anisotropic imperfections. By the analytic Fredholm theory, we transform these systems into the determination of the poles of two meromorphically continued inverse integral operator-valued functions $T^{-1}(w)$ and $T_\epsilon^{-1}(w)$ in the complex plane. Section 4 is devoted to the rigorous derivation of asymptotic expansions of the resonances as ϵ goes to zero.

2 Problem formulation

In this paper we consider two or three-dimensional bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) with C^2 -smooth boundary $\partial\Omega$, and let ν denotes the outward unit normal vector on $\partial\Omega$. We suppose that Ω contains a finite number m of (possibly anisotropic) bounded imperfections (see Figure 1). The geometry of each one is of the form $D_i = z_i + \epsilon B_i$, with a smooth and simply connected boundary ∂D_i , where $B_i \subset \mathbb{R}^n$ is a regular enough bounded domain representing the volume of the imperfection, $z_i \in \mathbb{R}^n$ is the vector position of its center and $\epsilon \in \mathbb{R}_+$ is the scale factor. The total collection of imperfections thus takes the form $D_\epsilon := \cup_{i=1}^m D_i$, with the following assumption:

$$\bar{D}_i \cap \bar{D}_j = \emptyset \quad \forall i \neq j, \quad \text{and } 0 < d_0 \leq \text{dist}(z_j, \partial\Omega), \quad (2.1)$$

where d_0 is a positive constant.

We assume that the constitutive parameters of the media in D_ϵ are represented by a real-valued symmetric matrix $\gamma_{D_\epsilon} = (\gamma_D^{ij}) \in C^1(D_\epsilon, \mathbb{R}^{n \times n})$ such that $\xi \cdot \gamma_{D_\epsilon}(x)\xi \geq a_0|\xi|^2 > 0$ for almost all $x \in D_\epsilon$ and all $\xi \in \mathbb{C}^n$, and where a_0 is a positive constant. Outside D_ϵ the background media is isotropic. We denote by A_ϵ the constitutive parameters of the anisotropic background \mathbb{R}^n given by

$$\gamma_\epsilon(x) := \begin{cases} \gamma_{D_\epsilon}(x) & x \in D_\epsilon \\ \gamma_1 I_n & x \in \Omega \setminus \bar{D}_\epsilon \\ \gamma_2 I_n & x \in \mathbb{R}^n \setminus \Omega \end{cases} \quad (2.2)$$

where I_n is the identity matrix in \mathbb{R}^n , and γ_1, γ_2 are two positive constants. For any regular function v and in terms of γ_ϵ , we have $\nabla \cdot \gamma_\epsilon \nabla v := \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (\gamma_\epsilon^{ij} \frac{\partial v}{\partial x_j})$.

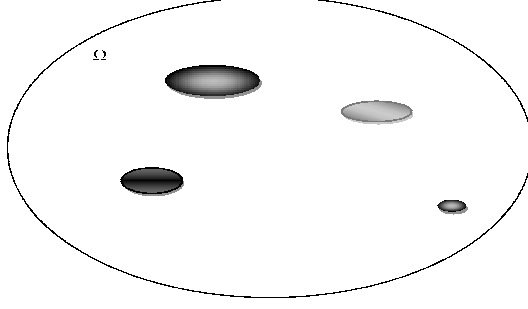


Figure 1: Configuration of the domain Ω with four imperfections ($m=4$).

We are interested in calculating resonances or scattering frequencies λ_ϵ of the following problem in dimension $n = 2$ or 3 :

$$\nabla \cdot \gamma_\epsilon \nabla u_\epsilon + \lambda_\epsilon^2 u_\epsilon = 0 \quad \text{in } \mathbb{R}^n \quad (2.3)$$

with $u_\epsilon \in H_{loc}^1(\mathbb{R}^n)$ satisfying the following radiation condition at infinity:

$$|\partial_r u_\epsilon - i \frac{\lambda_\epsilon}{\sqrt{\gamma_2}} u_\epsilon| \leq \frac{c}{r^{n-1}}.$$

where $r = |x|$, and the coefficient γ_ϵ takes on one value outside some bounded domain Ω , another inside Ω , and a different value inside the small (anisotropic) imperfections D_i in the interior of Ω .

Our goal is to express λ_ϵ in an asymptotic expansion in terms of the resonances of the unperturbed problem:

$$\nabla \cdot \gamma(x) \nabla u + \lambda^2 u = 0 \quad \text{in } \mathbb{R}^n \quad (2.4)$$

$$|\partial_r u - i \frac{\lambda}{\sqrt{\gamma_2}} u| \leq \frac{c}{r^{n-1}}.$$

where

$$\gamma(x) := \gamma_1 \chi(\Omega) I_n + \gamma_2 \chi(\mathbb{R}^n \setminus \bar{\Omega}) I_n, \quad (\text{for } \epsilon = 0),$$

where $\chi(\Omega)$ (resp. $\chi(\mathbb{R}^n \setminus \bar{\Omega})$) is the characteristic function of Ω (resp. of $\mathbb{R}^n \setminus \bar{\Omega}$), and γ_1 and γ_2 are given by (2.2).

To the best of our knowledge, this work is the first one to solve the resonance problems, in the presence of multiple (with finite number) anisotropic imperfections, but by referring to other existing related problems.

Now, we can remark that

Remark 2.4 1. For each $j = 1, \dots, m$, we put $\gamma_{D_\epsilon}(x) = \gamma_{D_j}(x)$ for all $x \in D_i$ and where $\gamma_{D_j} = (\gamma_{D_j}^{il})_{1 \leq i, l \leq n}$ is a symmetric matrix representing the constitutive parameters of the media in D_j are represented by a real-valued symmetric matrix $\gamma_{D_j} = (\gamma_{D_j}^{il}) \in C^1(D_j, \mathbb{R}^{n \times n})$ such that $\xi \cdot \gamma_{D_j}(x) \xi \geq a_j |\xi|^2 > 0$ for almost all $x \in D_j$ and all $\xi \in \mathbb{C}^n$, and where a_j is a positive constant such that $a_0 = \min_{1 \leq j \leq m} a_j$; where a_0 was given before.

2. For a function $u \in C^1(D_\epsilon)$ we define the conormal derivative by

$$\frac{\partial u}{\partial \nu_{\gamma_\epsilon}}(x) := \nu(x) \cdot \gamma_\epsilon(x) \nabla u(x), \quad x \in \partial D_\epsilon.$$

In terms of the previous notations, we have the following definition.

Definition 1 *Regarding (2.3), we may define:*

1) *The values of $\lambda_\epsilon \in C$ for which the following resonance problem*

$$\begin{cases} \nabla \cdot \gamma_\epsilon \nabla u_\epsilon + \lambda_\epsilon^2 u_\epsilon = 0 & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

has a nontrivial solution $u_\epsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ are called the resonant frequencies.

2) *The values of $\lambda_\epsilon \in C$ for which the following resonance problem*

$$\begin{cases} \nabla \cdot \gamma_\epsilon \nabla u_\epsilon + \lambda_\epsilon^2 u_\epsilon = 0, & \text{in } \mathbb{R}^n \setminus \bar{\Omega} \\ u_\epsilon = 0, & \text{on } \partial\Omega \\ |\partial_r u_\epsilon - i \frac{\lambda_\epsilon}{\sqrt{\gamma_2}} u_\epsilon| \leq \frac{C}{r^{n-1}}. \end{cases}$$

has a nontrivial solution u_ϵ are called the scattering frequencies.

The interior transmission eigenvalue problem (ITEP) corresponding to the scattering problem for an anisotropic inhomogeneous medium in \mathbb{R}^n , $n = 2, 3$ reads:

$$\begin{cases} \nabla \cdot \gamma_1 \nabla u + \lambda^2 u = 0 & \text{in } \Omega \setminus \bar{D}_\epsilon \\ \nabla \cdot \gamma_{D_\epsilon} \nabla u_\epsilon + \lambda_\epsilon^2 u_\epsilon = 0 & \text{in } D_\epsilon \\ u_\epsilon - u = h_1 & \text{on } \partial D_\epsilon \\ \frac{\partial u_\epsilon}{\partial \nu_{\gamma_\epsilon}} - \frac{\partial u}{\partial \nu} = h_2 & \text{on } \partial D_\epsilon \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.5)$$

with the given data $h_1 \in H^{1/2}(\partial D_\epsilon)$ and $h_2 \in H^{-1/2}(\partial D_\epsilon)$. The interior Neumann problem (on $\partial\Omega$) can be treated in essentially the same way as the interior Dirichlet problem.

Since γ_{D_ϵ} is symmetric, positive definite matrix, it follows that, in a well determined basis in \mathbb{R}^n , we can write $\gamma_{D_\epsilon}(x) := \text{diag}\{k_j(x), \quad j = 1, \dots, n\}$; where $k_j(x)$ ($x \in D_\epsilon$) is a positive valued-function. Consequently, by using Remark 2.4 we can write $\gamma_{D_i}(x) := \text{diag}\{k_j^i(x), \quad j = 1, \dots, n\}$, for $i = 1, \dots, m$, where $k_j^i(x)$ ($x \in D_i$) is a positive valued-function.

On the other hand, by assuming that k_j^i is constant on D_i , we may refer to the existing works (but for isotropic inclusions) [1, 3, 32] to define the polarization tensor, $M^{(i)}$, which is an $n \times n$, symmetric, positive definite matrix given by

$$M_{jl}^{(i)} = |B_i| \delta_{jl} + \left(\frac{\gamma(z_i)}{\text{tr}(\gamma_{D_i})} - 1 \right) \int_{\partial B_i} y_j \frac{\partial \phi_l^+}{\partial \nu} d\sigma_y \quad (2.6)$$

where $\text{tr}(\gamma_{D_i}) := \sum_{j=1}^n k_j^i$ means the trace of the matrix γ_{D_i} , $\gamma(z_i)$ is the background conductivity and the point z_i , and for $l = 1, \dots, m$; $\phi_l(y)$ is the unique function which satisfies

$$\begin{aligned} \Delta \phi_l &= 0 & \text{in } B_i \\ \Delta \phi_l &= 0 & \text{in } \mathbb{R}^n \setminus B_i \\ \gamma(z_i) \frac{\partial \phi_l^+}{\partial \nu} - \text{tr}(\gamma_{D_i}) \frac{\partial \phi_l^-}{\partial \nu} &= -\text{tr}(\gamma_{D_i}) \nu_l & \text{on } \partial B_i \end{aligned} \quad (2.7)$$

with ϕ_l continuous across ∂B_i and $\lim_{|y| \rightarrow \infty} \phi_l(y) = 0$. Here $\nu = (\nu_1, \dots, \nu_n)$ still denotes the outward unit normal to ∂B_i ; superscript $+$ and $-$ indicate the limiting values as we approach ∂B_i from outside B_i , and from inside B_i , respectively. Note that the polarization tensor depends on the conductivity, size, and shape of the imperfection.

It is known that the set of resonances $\{\lambda\}$ of the transmission problem (2.4) is discrete and symmetric about the imaginary axis if γ_1 and γ_2 are real. Further, it can be easily seen that all the resonances $\{\lambda\}$ are in the lower half-space $\text{Im}\lambda < 0$. They can be found explicitly for a sphere and a constant conductivity γ and are connected in this case with the zeros of certain Bessel functions. More elaborate results assert that for a strictly convex domain Ω in \mathbb{R}^3 and a constant conductivity $\gamma_1 > \gamma_2$ the resonances $\{\lambda\}$ accumulate on the real axis as $|\lambda| \rightarrow +\infty$. One would expect that for such a situation the resonances (at least for a sequence) come rapidly to the real axis as $|\lambda| \rightarrow +\infty$. This is in fact the case and was shown in a recent work by Popov and Vodev [23], [24], [25]. For a strictly convex domain it has been proved by Cardoso, Popov and Vodev [8] that if $\gamma_1 < \gamma_2$ there are no resonances in a strip above the real axis. We know of three methods in the literature for finding the resonances of the transmission problem. One is the method of the Rayleigh-Ritz type, well known in quantum chemistry [29]. Another closely related method consists of approximating the exact radiation condition by an appropriate boundary condition (Dirichlet condition or something like the on-surface boundary conditions [19], [16]) on a large sphere and then computing the eigenvalues of the resulting problem [12]. The strategy in the third method is to solve the time-dependent wave equation for any appropriate initial data at $t = 0$. The resonances are then calculated from the asymptotic expansion of the solution for fixed point x as $t \rightarrow +\infty$ [33]. Little is known about a constructive method for finding the resonances of the transmission problem like the variational principle for eigenvalues of the interior problems. For material on resonances of the scattering problem with Dirichlet or Neumann boundary condition on the boundary of the obstacle Ω we refer the reader for example to [6], [7], [20], [31], [30] and the references therein. For other important investigations concerning the interplay between resonances and geometry, we refer the readers to well-know works of Broer et al. [4, 5], and that of Zworski [34].

3 Integral equations and operator convergence

In this section will reformulate the problems (2.4) and (2.3) in terms of integral equation operators and show operator convergence. A similar integral equation method may be applied to solve the eigenvalue problems given by Definition 1 or the interior transmission eigenvalue problem (ITEP) defined in (2.5). The use of integral equations is a convenient tool for a number of investigations in scattering theory [10], [31], [22], [23], [25], [26], and for basic approaches by means of operators we may refer to [14]. We use it here to characterize the resonances as poles of a meromorphic operator-valued function on the whole complex plane. This characterization is the key point in deriving our asymptotic formulae.

3.1 Boundary integral formulation

Assume that $\gamma(x) := \gamma_1 \chi(\Omega) I_n + \gamma_2 \chi(\mathbb{R}^n \setminus \bar{\Omega}) I_n$, (for $\epsilon = 0$). It is well know that for ω not a resonance value for

$$(\gamma_2 \Delta + \omega^2),$$

there exists a unique Green's function $G^\omega(x, y)$ satisfying

$$(\gamma_2 \Delta_y + \omega^2) G^\omega(x, y) = -\delta_x(y) \quad (3.8)$$

along with the radiation condition

$$|\partial_r G^\omega - i \frac{\omega}{\sqrt{\gamma_2}} G^\omega| \leq \frac{C}{r^{n-1}}.$$

We actually know G^ω explicitly, when $n = 3$

$$G^\omega(x, y) = \frac{e^{i \frac{\omega}{\sqrt{\gamma_2}} |x-y|}}{4\pi \gamma_2 |x-y|}$$

and

$$G^\omega(x, y) = \frac{1}{\gamma_2} H_0^{(1)}\left(\frac{\omega}{\sqrt{\gamma_2}}|x - y|\right)$$

for $n = 2$. Here, $H_0^{(1)}$ is the Hankel function of order 0. By integration by parts, we see that if the pair $(u(x), \lambda)$ is a nontrivial solution to (2.4), outside of Ω $u(x)$ can be expressed as

$$u(x) = \int_{\partial\Omega} \gamma_2 \partial_{\nu_y} G^\lambda(x, y) u(y) d\sigma_y - \int_{\partial\Omega} \gamma_2 G^\lambda(x, y) \partial_\nu u(y)|_{\partial\Omega^+} d\sigma_y, \quad (3.9)$$

where the second term is continuous up to the boundary but the first is not. Define the operator N^ω to be the interior Dirichlet to Neumann map, i.e.

$$N^\omega : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega).$$

where

$$N^\omega(f) := \partial_\nu v|_{\partial\Omega^-}$$

for v the solution to

$$\begin{aligned} \gamma_1 \Delta v + \omega^2 v &= 0 \quad \text{in } \Omega \\ v &= f \quad \text{on } \partial\Omega. \end{aligned}$$

The above boundary value problem is well-posed in $H^1(\Omega)$ for any ω not a Dirichlet eigenvalue of the operator $\gamma_1 \Delta$ in the bounded domain Ω . Hence the operator N^ω is then well-defined on \mathbb{C} outside the set of these (real) eigenvalues.

Now, from the transmission condition

$$\gamma_1 \partial_\nu u|_{\partial\Omega^-} = \gamma_2 \partial_\nu u|_{\partial\Omega^+},$$

we get, by taking the limit of (3.9) as $x \rightarrow \partial\Omega^+$ (see Taylor [31] or Folland [13]),

$$(1 - \frac{\gamma_2}{2})u|_{\partial\Omega} = \gamma_2 \int_{\partial\Omega} \partial_\nu G^\lambda u|_{\partial\Omega} - \gamma_1 \int_{\partial\Omega} G^\lambda N^\lambda(u|_{\partial\Omega}).$$

If the single and double layer potential operators

$$S^\omega : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$$

and

$$D^\omega : H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$$

are defined by

$$S^\omega : g \mapsto \int_{\partial\Omega} G^\omega(x, y) g(y) d\sigma_y$$

and

$$D^\omega : f \mapsto \int_{\partial\Omega} \partial_{\nu_y} G^\omega(x, y) f(y) d\sigma_y,$$

we see that $u(x)|_{\partial\Omega}$ satisfies

$$\left((1 - \frac{\gamma_2}{2})I - \gamma_2 D^\lambda + \gamma_1 S^\lambda N^\lambda \right) (u|_{\partial\Omega}) = 0$$

where I is the identity operator. This suggests we define the operator T with complex parameter ω ,

$$T(\omega) : H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$$

by

$$T(\omega) = (1 - \frac{\gamma_2}{2})I - \gamma_2 D^\omega + \gamma_1 S^\omega N^\omega. \quad (3.10)$$

Then clearly the resonances of (2.4) are values of λ for which T has no inverse, since there exists $u \in H^{1/2}(\partial\Omega)$ such that $T(\lambda)u = 0$.

Let $\langle \cdot, \cdot \rangle$ be the $L^2(\partial\Omega)$ -inner product. By Green's formula it is easy to see that for any $u, v \in H^{1/2}(\partial\Omega)$

$$\langle N^\omega u, v \rangle = \langle u, (N^\omega)^* v \rangle$$

where the L^2 -adjoint operator of N^ω is given by

$$(N^\omega)^* = N^{\bar{\omega}}.$$

Moreover, one may check that we have

$$(S^\omega)^* : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$$

given by

$$(S^\omega)^* = S^{-\bar{\omega}},$$

and

$$(D^\omega)^* : H^{-1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$$

given by

$$(D^\omega)^* g = \int_{\partial\Omega} \partial_{\nu_x} G^{-\bar{\omega}}(x, y) g(y) d\sigma_y.$$

Hence the dual of T ,

$$T^*(\omega) : H^{-1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$$

is equal to

$$T^*(\omega) = (1 - \frac{\gamma_2}{2})I - \gamma_2(D^\omega)^* + \gamma_1 N^{\bar{\omega}} S^{-\bar{\omega}}.$$

Furthermore, from elliptic regularity it follows that

$$N^\omega - N^0 : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$$

is compact and so,

$$T(\omega) - T(0) : H^{1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$$

is also a compact operator. By the analytic Fredholm theory [27], [31] the resonances of (2.4) are poles of the meromorphically continued inverse $T^{-1}(\omega)$ to the whole complex plane if $n = 3$. This is also valid for $n = 2$, but then we have to continue $T^{-1}(\omega)$ to the Riemann surface for $\log \omega$ rather than \mathbb{C} .

Similarly, if the pair $(u_\epsilon, \lambda_\epsilon)$ is a nontrivial solution of (2.3), we can express u_ϵ in the exterior of Ω by

$$u_\epsilon(x) = \gamma_2 \int_{\partial\Omega} \partial_\nu G^{\lambda_\epsilon} u_\epsilon|_{\partial\Omega} - \gamma_2 \int_{\partial\Omega} G^{\lambda_\epsilon} \partial_\nu u_\epsilon|_{\partial\Omega+}.$$

Define now the interior Dirichlet to Neumann map corresponding to the medium with an inhomogeneity,

$$N_\epsilon^\omega : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$$

$$N_\epsilon^\omega(f) := \partial_\nu v_\epsilon|_{\partial\Omega-}$$

where v_ϵ is the solution to

$$\begin{aligned} \nabla \cdot \gamma_\epsilon \nabla v_\epsilon + \omega^2 v_\epsilon &= 0 \quad \text{in } \Omega \\ v_\epsilon &= f \quad \text{on } \partial\Omega. \end{aligned}$$

again, by taking the limit as $x \rightarrow \partial\Omega^-$, we see that $u_\epsilon|_{\partial\Omega}$ is a nontrivial solution to

$$\left((1 - \frac{\gamma_2}{2})I - \gamma_2 D^{\lambda_\epsilon} + \gamma_1 S^{\lambda_\epsilon} N_\epsilon^{\lambda_\epsilon} \right) (u_\epsilon|_{\partial\Omega}) = 0.$$

Now it seems natural to define

$$T_\epsilon(\omega) = (1 - \frac{\gamma_2}{2})I - \gamma_2 D^\omega + \gamma_1 S^\omega N_\epsilon^\omega, \quad (3.11)$$

so that the resonances of (2.3) are poles of the complex meromorphic $T_\epsilon^{-1}(\omega)$. We have thus reduced the analysis of the perturbed and unperturbed scattering problems to the asymptotic analysis of the poles of $T_\epsilon^{-1}(\omega)$ in a neighborhood of the poles of $T^{-1}(\omega)$.

3.2 Operator convergence

We will need to obtain pointwise convergence estimates for T_ϵ to T and a convenient asymptotic expansion in terms of ϵ . These operators are (constant permittivity) examples of those described in [2] with the exception that in that paper the operators correspond to the special case $\gamma_2 = 1$. We also mention that the authors did not emphasize the dependence of the operators on ω since the frequency was assumed to be fixed throughout that work. Since

$$T_\epsilon(\omega) - T(\omega) = \gamma_1 S^\omega (N_\epsilon^\omega - N^\omega),$$

the difference of the operators here is just a constant multiple of the difference of the operators in Proposition 1 of [2]. We may use the following definition.

Definition 2 (*Collectively compact*) A set of operators, $\{K_p; \ p \in I\}$, is collectively compact iff the set $K_p(B_0) := \{K_p(x); \ x \in B_0\}$ is totally bounded (or has compact closure). Here, I is a set of indexes and B_0 is the unit ball in \mathbb{R}^n .

We reformulate Proposition 1 of [2] here in our context. Its proof depends on the work of Vogelius and Volkov [32].

Proposition 1 Let λ be a resonance value of (2.4). Assume that γ_{D_i} , $i = 1, \dots, m$ be given as in Section 2. Let $T_\epsilon(\omega)$ be defined by (3.11) and $T(\omega)$ by (3.10). Then there exists some neighborhood B_λ around λ such that for $\omega \in B_\lambda$ we have the following

- (a) $T_\epsilon(\omega)$ converges to $T(\omega)$ pointwise.
- (b) For each fixed ω , there exists ϵ_0 such that the set $\{T_\epsilon(\omega) - T(\omega); \ \epsilon < \epsilon_0\}$ is collectively compact.
- (c) For each fixed $\omega \in B_\lambda \setminus \{\lambda\}$ there exists a constant C that is independent of ϵ such that for any $f \in H^{1/2}(\partial\Omega)$, $T_\epsilon(\omega)^{-1}$ exists and

$$\|T_\epsilon(\omega)^{-1}f\|_{H^{1/2}(\partial\Omega)} \leq C_\omega \|f\|_{H^{1/2}(\partial\Omega)}.$$

- (d) The following asymptotic formula holds

$$\begin{aligned} (T(\omega) - T_\epsilon(\omega))(f)(x) &= \gamma_1 S^\omega (N^\omega - N_\epsilon^\omega)(f)(x) \\ &= -\epsilon^n \sum_{i=1}^m \gamma_1 \left(1 - \frac{\gamma_1}{\text{tr}(\gamma_{D_i})}\right) \nabla v(z_i) \cdot M^{(i)} \nabla_y G(x, z_i) + o(\epsilon^n) \end{aligned} \tag{3.12}$$

where v is the solution to

$$\begin{aligned} \nabla \cdot \gamma \nabla v + \omega^2 v &= 0 \quad \text{in } \Omega \\ v &= f \quad \text{on } \partial\Omega; \end{aligned} \tag{3.13}$$

and the asymptotic term $o(\epsilon^n)$ (and its derivatives) are uniform in $x \in \partial\Omega$ and $\omega \in B_\lambda$.

One also gets the following estimate on pointwise convergence of the operators and their dual:

Corollary 1 Let λ be a resonance value of (2.4). $T_\epsilon(\omega)$ be defined by (3.11) and $T(\omega)$ by (3.10). Then there exists some neighborhood B_λ around λ such that for $f \in H^{1/2}(\partial\Omega)$, $g \in H^{-1/2}(\partial\Omega)$, and $\omega \in B_\lambda$ we have

$$\|(T_\epsilon(\omega) - T(\omega))f\|_{H^{1/2}(\partial\Omega)} \leq C\epsilon^n \tag{3.14}$$

and

$$\|(T_\epsilon^*(\omega) - T^*(\omega))g\|_{H^{-1/2}(\partial\Omega)} \leq C\epsilon^n \tag{3.15}$$

where C is independent of ϵ and uniform for ω in B_λ .

Proof The first estimate follows from (d) of Proposition 1. For the difference of the dual operators,

$$T_\epsilon^*(\omega) - T^*(\omega) = \gamma_1 (N_\epsilon^{\bar{\omega}} - N^{\bar{\omega}}) S^{-\bar{\omega}}$$

by the same proof as in [2] we have a similar asymptotic formula as above, from which the pointwise convergence estimate follows. □

4 Asymptotic formulae for the resonances

Let λ be a pole of T , i.e. $T(\lambda)^{-1}$ does not exist. It is known, then [17], [28] that for each integer k , the null space of $N(T(\lambda)^k)$ is finite dimensional, and that for δ small enough, $T^{-1}(z)$ is an analytic operator in $B_\delta(\lambda) \setminus \{\lambda\}$. Let α be the ascent of T , i.e. α is the smallest integer such that

$$N(T(\lambda)^\alpha) = N(T(\lambda)^{\alpha+1}).$$

By [17], [28] such a smallest integer exists and is bigger than or equal to one. It is the order of λ as a pole of $T^{-1}(z)$ in B_δ . Then the algebraic multiplicity p of λ is defined by

$$p = \dim N(T(\lambda)^\alpha).$$

The geometric multiplicity

$$m = \dim N(T)$$

does not clearly not exceed the algebraic multiplicity p , and since

$$N(T(\lambda)^{\alpha-1}) \neq N(T(\lambda)^\alpha),$$

both multiplicities coincide if and only if the order α of λ is equal to one [15]. It is also known that for any δ small enough, there exists ϵ_0 such that for any $\epsilon < \epsilon_0$, T_ϵ has exactly m (resp. p) resonances $\{\lambda_\epsilon^j\}$, counted according to geometric (resp. algebraic) multiplicity, in $B_\delta(\lambda)$ and such that $T_\epsilon^{-1}(z)$ exists on $\Gamma = \partial B_\delta(\lambda)$. Define

$$E = \frac{1}{2\pi i} \int_\Gamma T^{-1}(z) dz, \quad (4.16)$$

as the projection of $H^{1/2}(\partial\Omega)$ onto the generalized “eigenspace” $N(T(\lambda)^\alpha)$ along $R(T(\lambda)^\alpha)$. Similarly, define

$$E_\epsilon = \frac{1}{2\pi i} \int_\Gamma T_\epsilon^{-1}(z) dz$$

which is the projection onto the direct sum of the generalized eigenspaces of λ_ϵ^j . We know [17] that for ϵ small enough,

$$p = \dim N(T(\lambda_\epsilon)^\alpha)$$

and that for any $u \in R(E) = N(T(\lambda)^\alpha)$, there exists, for each ϵ , a $u_\epsilon \in R(E_\epsilon) = N(T(\lambda_\epsilon)^\alpha)$ such that

$$\|u_\epsilon - u\|_{H^{1/2}(\partial\Omega)} \rightarrow 0,$$

i.e.,

$$\delta(R(E), R(E_\epsilon)) \rightarrow 0.$$

where δ is the distance between the unit balls of the two subspaces. Also, since $T^{-1}(z)$ is a finitely meromorphic operator in $B_\delta(\lambda)$ (for δ small enough), by for example [27] we know that $T^{-1}(z)$ has the expansion

$$T^{-1}(z) = \frac{L_\alpha}{(z - \lambda)^\alpha} + \frac{L_{\alpha-1}}{(z - \lambda)^{\alpha-1}} + \dots + L_0(z) \quad (4.17)$$

where for $k = 1, \dots, \alpha$ each L_k is defined by

$$L_k = -T^{k-1}E$$

and $L_0(z)$ is some analytic operator in $\mathcal{L}(H^{1/2}(\partial\Omega))$.

We search now for λ_ϵ , a pole of $T_\epsilon^{-1}(z)$ in $B_\delta(\lambda)$. To this end, we recall that we know such a pole exists. Then there exists and $L^2(\partial\Omega)$ -normalized $u_\epsilon \in N(T_\epsilon)$ such that

$$T_\epsilon(\lambda_\epsilon)u_\epsilon = 0,$$

and hence

$$T(\lambda_\epsilon)u_\epsilon + (T_\epsilon(\lambda_\epsilon) - T(\lambda_\epsilon))u_\epsilon = 0. \quad (4.18)$$

Assuming that $\lambda_\epsilon \neq \lambda$, $T(\lambda_\epsilon)^{-1}$ exists and

$$u_\epsilon + T(\lambda_\epsilon)^{-1}(T_\epsilon(\lambda_\epsilon) - T(\lambda_\epsilon))u_\epsilon = 0. \quad (4.19)$$

Taking the L^2 -inner product with u_ϵ , we obtain

$$1 + \langle T(\lambda_\epsilon)^{-1}(T_\epsilon(\lambda_\epsilon) - T(\lambda_\epsilon))u_\epsilon, u_\epsilon \rangle = 0. \quad (4.20)$$

λ_ϵ is a resonance of the transmission problem in $B_\delta(\lambda)$ if and only if there exists

$$u_\epsilon \in H^{1/2}(\partial\Omega)$$

with

$$\langle u_\epsilon, u_\epsilon \rangle = \|u_\epsilon\|_{L^2(\partial\Omega)}^2 = 1$$

such that λ_ϵ is a root of the complex meromorphic function

$$g(z) = 1 + \langle T(z)^{-1}(T_\epsilon(z) - T(z))u_\epsilon, u_\epsilon \rangle. \quad (4.21)$$

From now on we concentrate for simplicity on resonances that are simple, i.e. with $\alpha = 1$. Note that in this case m does not necessarily equal one, but $p = m$ and $R(E) = N(T)$. The derivation of an asymptotic formula in the general case follows from similar arguments and will be presented at the end of this paper. We will need the following lemma:

Lemma 1 *Assume $\alpha = 1$. Let $\{u^j\}_{j=1}^m$ be an L^2 -orthonormal basis of eigenfunctions for $E = N(T(\lambda))$, and let $\{u^{j*}\}_{j=1}^m$ be the dual basis for $E^* = N(T^*(\lambda))$, that is, with*

$$\langle u^j, u^{i*} \rangle = \delta_{ij}.$$

Then let $u_\epsilon^j \in N(T_\epsilon(\lambda_\epsilon^j))$ be chosen such that

$$\|u_\epsilon^j - u^j\|_{H^{1/2}(\partial\Omega)} \rightarrow 0.$$

Then for $z = \lambda_\epsilon^j$,

$$\begin{aligned} \langle L_1(T - T_\epsilon)u_\epsilon^j, u_\epsilon^j \rangle &= \langle (T_\epsilon - T)u^j, u^{j*} \rangle \\ &+ o(\|(T - T_\epsilon)|_{R(E)}\|) + o(\|(T - T_\epsilon)^*|_{R(E^*)}\|). \end{aligned} \quad (4.22)$$

Proof First note that

$$\begin{aligned} \langle L_1(T - T_\epsilon)u_\epsilon^j, u_\epsilon^j \rangle &= \langle (T - T_\epsilon)u_\epsilon^j, L_1^*u_\epsilon^j \rangle \\ &= \sum_i \langle (T - T_\epsilon)u_\epsilon^j, u^{i*} \rangle \langle u_\epsilon^j, u^{i*} \rangle. \end{aligned}$$

By adding and subtracting u^j to u_ϵ^j , we obtain

$$\begin{aligned} \langle L_1(T - T_\epsilon)u_\epsilon^j, u_\epsilon^j \rangle &= \sum_i \langle (T_\epsilon - T)(u_\epsilon^j - u^j), u^{i*} \rangle \langle u^{i*}, u_\epsilon^j \rangle \\ &+ \sum_i \langle (T_\epsilon - T)u^j, u^{i*} \rangle \langle u^{i*}, u_\epsilon^j \rangle \\ &= \sum_i \langle (u_\epsilon^j - u^j), (T_\epsilon - T)^*u^{i*} \rangle \langle u^{i*}, u_\epsilon^j \rangle \\ &+ \sum_i \langle (T_\epsilon - T)u^j, u^{i*} \rangle \langle u^{i*}, u_\epsilon^j \rangle \end{aligned} \quad (4.23)$$

which, by the orthogonality of the basis functions and the convergence of u_ϵ^j , proves the lemma. \square

Using the above lemma, we may prove the following result.

Proposition 2 *Let T and T_ϵ be the operators defined by (3.10) and (3.11) respectively. Let λ be a resonance of T of order one and multiplicity m . For δ and ϵ small enough, T_ϵ has exactly m resonances (λ_ϵ^j) , counted according to multiplicity in $B_\delta(\lambda)$ and the following asymptotic formula holds:*

$$\begin{aligned} \frac{1}{m} \sum_{j=1}^m \lambda_\epsilon^j - \lambda &= \frac{1}{m} \sum_{j=1}^m \langle (T_\epsilon(\lambda_\epsilon^j) - T(\lambda_\epsilon^j))u^j, u^{j*} \rangle \\ &+ o(\|(T - T_\epsilon)|_{R(E)}\|) + o(\|(T - T_\epsilon)^*|_{R(E^*)}\|), \end{aligned}$$

where E is the projection operator defined by (4.16), E^* its dual, and $\{u^j, u^{j*}\}_{j=1}^m$ are mutually L^2 -orthonormal bases of eigenfunctions of $T(\lambda)$ and $T^*(\lambda)$.

Proof Let the function g be defined by (4.21). Using the expansion (4.17), we may rewrite $g(z)$ as

$$\begin{aligned} g(z) &= 1 + \frac{\langle L_\alpha(T_\epsilon(z) - T(z))u_\epsilon, u_\epsilon \rangle}{(z - \lambda)^\alpha} \\ &+ \frac{\langle L_{\alpha-1}(T_\epsilon(z) - T(z))u_\epsilon, u_\epsilon \rangle}{(z - \lambda)^{\alpha-1}} \\ &+ \dots + \langle L_0(z)(T_\epsilon(z) - T(z))u_\epsilon, u_\epsilon \rangle. \end{aligned}$$

Since

$$(\lambda_\epsilon - \lambda)^\alpha g(\lambda_\epsilon) = 0,$$

this gives us

$$\begin{aligned} (\lambda_\epsilon - \lambda)^\alpha &= \langle L_\alpha(T(\lambda_\epsilon) - T_\epsilon(\lambda_\epsilon))u_\epsilon, u_\epsilon \rangle \\ &+ (\lambda_\epsilon - \lambda) \langle L_{\alpha-1}(T(\lambda_\epsilon) - T_\epsilon(\lambda_\epsilon))u_\epsilon, u_\epsilon \rangle \\ &+ \dots + (\lambda_\epsilon - \lambda)^\alpha \langle L_0(\lambda_\epsilon)(T(\lambda_\epsilon) - T_\epsilon(\lambda_\epsilon))u_\epsilon, u_\epsilon \rangle. \end{aligned}$$

Bringing the last term over to the left hand side,

$$\begin{aligned} (\lambda_\epsilon - \lambda)^\alpha [1 - \langle L_0(T - T_\epsilon)u_\epsilon, u_\epsilon \rangle] &= \langle L_\alpha(T - T_\epsilon)u_\epsilon, u_\epsilon \rangle \\ &+ \dots + (\lambda_\epsilon - \lambda)^{\alpha-1} \langle L_1(T - T_\epsilon)u_\epsilon, u_\epsilon \rangle \end{aligned} \quad (4.24)$$

where all operators are evaluated at $z = \lambda_\epsilon$. Now define the analytic function

$$h_\epsilon(z) = \langle L_0(z)(T(z) - T_\epsilon(z))u_\epsilon, u_\epsilon \rangle.$$

We know, from the pointwise convergence of T_ϵ to T and the $H^{1/2}$ convergence of u_ϵ to u , that $h_\epsilon(z) \rightarrow 0$ in B_δ , so

$$\frac{1}{1 - h_\epsilon(z)} = 1 + h_\epsilon(z) + h_\epsilon^2(z) + \dots \quad (4.25)$$

where the expansion is uniform in B_δ for ϵ small enough. Using Lemma 1, (4.25), (4.24), and averaging over j , the proof holds. \square

The following Lemma describes the dual eigenspace.

Lemma 2 *For $j = 1, \dots, m$, we may choose*

$$u^{j*} = c_{jk}(S^{-\bar{\lambda}})^{-1}(\overline{u^k})$$

where the coefficients are given by

$$c_{jk} = (A^{-1})_{jk}$$

for

$$A_{ki} = \langle (S^{-\bar{\lambda}})^{-1}(\overline{u^k}), u^i \rangle.$$

Proof First we show that for any $j = 1, \dots, m$,

$$(S^{-\bar{\lambda}})^{-1}(\bar{u}^j) \in N(T^*(\lambda)).$$

Observe that

$$S^{-\bar{\lambda}}T^*(S^{-\bar{\lambda}})^{-1}(\bar{u}^j) = (1 - \frac{\gamma_2}{2})\bar{u}^j - \gamma_2 S^{-\bar{\lambda}}(D^\lambda)^*(S^{-\bar{\lambda}})^{-1}(\bar{u}^j) + \gamma_1 S^{-\bar{\lambda}}N^{\bar{\lambda}}(\bar{u}^j).$$

From $T(\lambda)(u^j) = 0$ it follows that

$$S^{-\bar{\lambda}}T^*(S^{-\bar{\lambda}})^{-1}(\bar{u}^j) = \gamma_2(D^{-\bar{\lambda}} - S^{-\bar{\lambda}}(D^\lambda)^*(S^{-\bar{\lambda}})^{-1})(\bar{u}^j).$$

The Calderón's commutation relation $SD = D^*S$ yields

$$S^{-\bar{\lambda}}T^*(S^{-\bar{\lambda}})^{-1}(\bar{u}^j) = 0,$$

and hence

$$T^*(S^{-\bar{\lambda}})^{-1}(\bar{u}^j) = 0.$$

In addition, for

$$u^{j*} = c_{jk}(S^{-\bar{\lambda}})^{-1}(\bar{u}^k),$$

we have the desired orthogonality properties

$$\langle u^{j*}, u^i \rangle = \delta_{ij}$$

□

On the other hand, we may compute the first term on the right hand side of Proposition 2 to prove the following main result.

Theorem 1 *Assume that we have (2.1), and all hypothesis of Proposition 2. Let γ_{D_i} be given as in Section 2. Then, the following asymptotic formula holds:*

$$\frac{1}{m} \sum_{j=1}^m \lambda_\epsilon^j - \lambda = -\epsilon^n \frac{1}{m} \sum_{i=1}^m \gamma_1 \left(1 - \frac{\gamma_1}{\text{tr}(\gamma_{D_i})}\right) \sum_{j,l=1}^m \nabla u^j(0) \cdot M^{(i)} \nabla c_{jl} u^l(0) + o(\epsilon^n)$$

where the coefficients are given by

$$c_{jl} = (A^{-1})_{jl}$$

for

$$A_{li} = \langle (S^{-\bar{\lambda}})^{-1}(\bar{u}^l), u^i \rangle.$$

Proof Let us now compute the first term on the right hand side of Proposition 2. Consider one term in the summation corresponding to a particular j . For clarity of exposition we will temporarily neglect the j superscript on λ_ϵ , u , and u^* . We have

$$\begin{aligned} \langle (T_\epsilon(\lambda_\epsilon) - T(\lambda_\epsilon))u, u^* \rangle &= \langle \gamma_1 S^{\lambda_\epsilon} (N_\epsilon^{\lambda_\epsilon} - N^{\lambda_\epsilon})u, u^* \rangle \\ &= \langle \gamma_1 (N_\epsilon^{\lambda_\epsilon} - N^{\lambda_\epsilon})u, S^{-\bar{\lambda}_\epsilon} u^* \rangle \\ &= \int_{\partial\Omega} \gamma_1 (N_\epsilon^{\lambda_\epsilon} - N^{\lambda_\epsilon})u(x) \overline{S^{-\bar{\lambda}_\epsilon} u^*(x)} d\sigma_x. \end{aligned}$$

Recall that

$$N^{\lambda_\epsilon} u = \frac{\partial \alpha^\epsilon}{\partial \nu}$$

and

$$N_\epsilon^{\lambda_\epsilon} u = \frac{\partial \beta^\epsilon}{\partial \nu}$$

where α_ϵ solves

$$\begin{aligned}\gamma_1 \Delta \alpha^\epsilon + (\lambda_\epsilon)^2 \alpha^\epsilon &= 0 & \text{in } \Omega \\ \alpha^\epsilon &= u & \text{on } \partial\Omega\end{aligned}$$

and β^ϵ solves

$$\begin{aligned}\nabla \cdot \gamma_\epsilon \nabla \beta^\epsilon + (\lambda_\epsilon)^2 \beta^\epsilon &= 0 & \text{in } \Omega \\ \beta^\epsilon &= u & \text{on } \partial\Omega.\end{aligned}$$

From (3.12), we have the uniform asymptotic expansion for y on $\partial\Omega$,

$$\frac{\partial \beta^\epsilon}{\partial \nu} - \frac{\partial \alpha^\epsilon}{\partial \nu} = -\epsilon^n \sum_{i=1}^m \left(1 - \frac{\gamma_1}{\text{tr}(\gamma_{D_i})}\right) \nabla \alpha^\epsilon(0) \cdot M^{(i)} \frac{\partial}{\partial \nu_y} \nabla_x \hat{N}^{(\lambda_\epsilon)}(0, y) + o(\epsilon^n). \quad (4.26)$$

where we have now noted the dependence of \hat{N} on ω explicitly. Since $\lambda_\epsilon \rightarrow \lambda$ and u satisfies $\gamma_1 \Delta u + \lambda^2 u = 0$ in Ω we can show that

$$|\nabla \alpha^\epsilon(0) - \nabla u(0)| \rightarrow 0.$$

Therefore we obtain

$$\frac{\partial \beta^\epsilon}{\partial \nu} - \frac{\partial \alpha^\epsilon}{\partial \nu} = -\epsilon^n \sum_{i=1}^m \left(1 - \frac{\gamma_1}{\text{tr}(\gamma_{D_i})}\right) \nabla u(0) \cdot M^{(i)} \frac{\partial}{\partial \nu_y} \nabla_x \hat{N}^{(\lambda_\epsilon)}(0, y) + o(\epsilon^n). \quad (4.27)$$

Define $w_\epsilon \in H^1(\Omega)$ by

$$\begin{aligned}\gamma_1 \Delta w_\epsilon + \bar{\lambda} w_\epsilon &= 0 & \text{in } \Omega \\ w_\epsilon &= S^{-\bar{\lambda}_\epsilon} u^* & \text{on } \partial\Omega.\end{aligned}$$

Then by using (4.27) we deduce that

$$\begin{aligned}\langle (N_\epsilon^{\lambda_\epsilon} - N^{\lambda_\epsilon})u, w_\epsilon \rangle &= \left\langle \frac{\partial \beta^\epsilon}{\partial \nu} - \frac{\partial \alpha^\epsilon}{\partial \nu}, w_\epsilon \right\rangle \\ &= -\epsilon^n \sum_{i=1}^m \left(1 - \frac{\gamma_1}{\text{tr}(\gamma_{D_i})}\right) \nabla u(0) \cdot M^{(i)} \nabla \bar{w}(0) \\ &\quad + o(\epsilon^n)\end{aligned}$$

where w is the H^1 limit of w_ϵ , that is, (returning to using the j superscript), w^j satisfies

$$\begin{aligned}\gamma_1 \Delta w^j + \bar{\lambda} w^j &= 0 & \text{in } \Omega \\ w^j &= S^{-\bar{\lambda}} u^{j*} & \text{on } \partial\Omega.\end{aligned}$$

Lemma 2 gives that

$$\bar{w}^j = c_{jk} u^k,$$

from which we obtain the next theorem. □

In the case where the resonance λ is not simple ($\alpha > 1$), we can easily generalize this result to the following theorem.

Theorem 2 *Let T and T_ϵ be the operators defined by (3.10) and (3.11) respectively. Let λ be a resonance of T of order (ascent) α , geometric multiplicity m and algebraic multiplicity p . Let $\{u^j\}$ be an $L^2(\partial\Omega)$ -orthonormal basis for $N(T)$. Then for δ and ϵ small enough, T_ϵ has exactly*

m resonances $\{\lambda_\epsilon^j\}$, counted according to geometric multiplicity in $B_\delta(\lambda)$. For $j = 1 \dots m$ the following asymptotic formula holds:

$$\begin{aligned} (\lambda_\epsilon^j - \lambda)^\alpha &= -\epsilon^n \sum_{i,l=1}^m \gamma_1 \left(1 - \frac{\gamma_1}{\text{tr}(\gamma_{D_i})}\right) \nabla u^j(0) \cdot M^{(i)} \nabla c_{jl} u^l(0) \\ &+ o(\epsilon^n), \end{aligned}$$

where the coefficients are given by

$$c_{jl} = (A^{-1})_{jl}$$

for

$$A_{li} = \langle (S^{-\bar{\lambda}})^{-1}(\bar{u}^l), u^i \rangle.$$

Proof From the same proof as Lemma 1, we can show that for any $k = 1, \dots, \alpha$,

$$\begin{aligned} \langle L_k(T - T_\epsilon)u_\epsilon^j, u_\epsilon^j \rangle &= \langle (T - T_\epsilon)u^j, u^{j*} \rangle \\ &+ o(\|T - T_\epsilon|_{R(E)}\|) + o(\|(T - T_\epsilon)^*|_{R(E^*)}\|). \end{aligned}$$

For $k = 1, \dots, \alpha - 1$, it also follows that

$$(\lambda_\epsilon^j - \lambda)^{\alpha-k} \langle L_k(T - T_\epsilon)u_\epsilon^j, u_\epsilon^j \rangle = o(\|T - T_\epsilon|_{R(E)}\|) + o(\|(T - T_\epsilon)^*|_{R(E^*)}\|).$$

Using this fact with (4.25) and (4.24) to obtain that

$$(\lambda_\epsilon^j - \lambda)^\alpha = \langle (T - T_\epsilon)u^j, u^{j*} \rangle + o(\epsilon^n)$$

which achieves the proof of the theorem. \square

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